

Energy identity for a class of approximate biharmonic maps into sphere in dimension four

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Abstract

We consider in dimension four weakly convergent sequences of approximate biharmonic maps into sphere with bi-tension fields bounded in L^p for some $p > 1$. We prove an energy identity that accounts for the loss of Hessian energies by the sum of Hessian energies over finitely many nontrivial biharmonic maps on \mathbb{R}^4 .

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ be the unit sphere. Recall the Sobolev space $W^{l,p}(\Omega, \mathbb{S}^k)$, $1 \leq l < +\infty$ and $1 \leq p < +\infty$, is defined by

$$W^{l,p}(\Omega, \mathbb{S}^k) = \left\{ v \in W^{l,p}(\Omega, \mathbb{R}^{k+1}) : v(x) \in \mathbb{S}^k \text{ a.e. } x \in \Omega \right\}.$$

In this paper we will discuss the limiting behavior of weakly convergent sequences of approximate (extrinsic) biharmonic maps $\{u_m\} \subset W^{2,2}(\Omega, \mathbb{S}^k)$ in dimension $n = 4$, especially an energy identity and oscillation convergence. First we recall the notion of approximate (extrinsic) biharmonic maps.

Definition 1.1 *A map $u \in W^{2,2}(\Omega, \mathbb{S}^k)$ is called an approximate biharmonic map if there exists a bi-tension field $h \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{k+1})$ such that*

$$\Delta^2 u + (|\Delta u|^2 + \Delta|\nabla u|^2 + 2\langle \nabla u, \nabla \Delta u \rangle) u = h \quad (1.1)$$

in the distribution sense. In particular, if $h = 0$ then u is called a biharmonic map to \mathbb{S}^k .

Note that biharmonic maps to \mathbb{S}^k are critical points of the Hessian energy functional

$$E_2(u) = \int_{\Omega} |\nabla^2 u|^2 dx$$

over $W^{2,2}(\Omega, \mathbb{S}^k)$. Biharmonic maps are higher order extensions of harmonic maps. The study of regularity of biharmonic maps has generated considerable interests after the initial work by Chang-Wang-Yang [2], the readers can refer to Wang [20, 21, 22], Strzelecki [18], Lamm-Rivierè [11], Struwe [19], Scheven [16, 17] (see also Ku [9] and Gong-Lamm-Wang [5] for the boundary regularity). In particular, the interior regularity theorem asserts that the smoothness of $W^{2,2}$ -biharmonic maps holds in dimension $n = 4$, and the partial regularity of stationary $W^{2,2}$ -biharmonic maps holds in dimensions $n \geq 5$.

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It is an important observation that biharmonic maps are invariant under dilations in \mathbb{R}^n for $n = 4$. Such a property leads to non-compactness of biharmonic maps in dimension 4, which prompts recent studies by Wang [20] and Hornung-Moser [7] concerning the failure of strong convergence for weakly convergent biharmonic maps. Roughly speaking, the results in [20] and [7] assert that the failure of strong convergence occurs at finitely many concentration points of Hessian energy, where finitely many bubbles (i.e. nontrivial biharmonic maps on \mathbb{R}^4) are generated, and the total Hessian energies from these bubbles account for the total loss of Hessian energies during the process of convergence.

Our first result is to extend the results from [20] and [7] to the context of suitable approximate biharmonic maps to \mathbb{S}^k . More precisely, we have

Theorem 1.2 *For $n = 4$, suppose $\{u_m\} \subset W^{2,2}(\Omega, \mathbb{S}^k)$ is a sequence of approximate biharmonic maps, which are bounded in $W^{2,2}(\Omega, \mathbb{S}^k)$ and have their bi-tension fields h_m bounded in L^p for some $p > 1$, i.e.*

$$M := \sup_m \left(\|u_m\|_{W^{2,2}} + \|h_m\|_{L^p} \right) < +\infty. \quad (1.2)$$

Assume $u_m \rightharpoonup u$ in $W^{2,2}$ and $h_m \rightharpoonup h$ in L^p . Then

- (i) u is an approximate biharmonic map to \mathbb{S}^k with h as its bi-tension field.
- (ii) There exist a nonnegative integer L depending only on M and $\{x_1, \dots, x_L\} \subset \Omega$ such that $u_m \rightarrow u$ strongly in $W_{\text{loc}}^{2,2} \cap C_{\text{loc}}^0(\Omega \setminus \{x_1, \dots, x_L\}, \mathbb{S}^k)$.
- (iii) For $1 \leq i \leq L$, there exist a positive integer L_i depending only on M and nontrivial smooth biharmonic map ω_{ij} from \mathbb{R}^4 to \mathbb{S}^k with finite Hessian energy, $1 \leq j \leq L_i$, such that

$$\lim_{m \rightarrow \infty} \int_{B_{r_i}(x_i)} |\nabla^2 u_m|^2 = \int_{B_{r_i}(x_i)} |\nabla^2 u|^2 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla^2 \omega_{ij}|^2, \quad (1.3)$$

and

$$\lim_{m \rightarrow \infty} \int_{B_{r_i}(x_i)} |\nabla u_m|^4 = \int_{B_{r_i}(x_i)} |\nabla u|^4 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla \omega_{ij}|^4, \quad (1.4)$$

where $r_i = \frac{1}{2} \min_{1 \leq j \leq L, j \neq i} \{|x_i - x_j|, \text{dist}(x_i, \partial\Omega)\}$.

The idea to prove Theorem 1.2 is based on the duality between the Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$ (see §2 below for the definitions and basic properties). More precisely, we can bound the $L^{2,1}$ -norm of $\nabla^2 u_m$ in the neck region, while showing the $L^{2,\infty}$ -norm of $\nabla^2 u_m$ can be arbitrarily small in the neck region. Our argument to estimate $\|\nabla^2 u_m\|_{L^{2,1}}$ relies heavily on the symmetry of \mathbb{S}^k - a property that was earlier utilized by [2], [20], [18], and [9] in the study of biharmonic maps. However, the argument to establish the estimation of $\|\nabla^2 u_m\|_{L^{2,\infty}}$ does not utilize the symmetry of \mathbb{S}^k and hence holds for any target manifold N .

We conjecture that Theorem 1.1 remains to be true for any target manifold N . For a general target manifold N , it remains to be a difficult question on how to obtain $L^{2,1}$ -estimate for $\nabla^2 u$ similar to (2.3) in Theorem 2.3. In a forthcoming paper [24], we will employ a different approach similar to [7] to prove Theorem 1.1 under the stronger assumption that the bi-tension fields are bounded in L^p for some $p > \frac{4}{3}$.

We would like to remark that the corresponding $L^{2,1}$ -estimate of ∇u plays a very important role in the study of harmonic maps by Hélein [6]. Later, Lin-Rivière [13] (and Lin-Wang [14] respectively) utilized the duality between $L^{2,1}$ and $L^{2,\infty}$ to study the energy quantization effect of

harmonic maps (and approximate harmonic maps respectively) to \mathbb{S}^k in higher dimensions. See also Laurain-Riviere [12] for some most recent related works.

From the view of point for applications, Theorem 1.2 can be a useful extension of the results by [20] and [7]. A typical application of Theorem 1.2 is to study asymptotic behavior at time infinity for the heat flow of biharmonic maps in dimension 4.

Let's review some recent studies on the heat flow of biharmonic maps undertaken by Lamm [10], Gastel [4], Wang [23], and Moser [15]. For a given compact Riemannian manifold $N \subset \mathbb{R}^{k+1}$ without boundary, the equation of heat flow of (extrinsic) biharmonic maps into N is to seek $u : \Omega \times [0, +\infty) \rightarrow N$ that solves (see Lamm [10]):

$$u_t + \Delta^2 u = \Delta(\mathbb{B}(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla(\mathbb{P}(u)) \rangle - \langle \Delta(\mathbb{P}(u)), \Delta u \rangle, \quad \Omega \times (0, +\infty) \quad (1.5)$$

$$u = u_0, \quad \Omega \times \{0\} \quad (1.6)$$

$$(u, \frac{\partial u}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu}), \quad \partial\Omega \times (0, +\infty), \quad (1.7)$$

where $u_0 \in W^{2,2}(\Omega, N)$ is a given map, $\mathbb{P}(y) : \mathbb{R}^{k+1} \rightarrow T_y N$ is the orthogonal projection from \mathbb{R}^{k+1} to the tangent space of N at $y \in N$, and $\mathbb{B}(y)(X, Y) = -\nabla_X \mathbb{P}(y)(Y)$, $\forall X, Y \in T_y N$, is the second fundamental form of $N \subset \mathbb{R}^{k+1}$. Note that any t -independent solution $u : \Omega \rightarrow N$ of (1.5) is a biharmonic map to N .

In dimension $n = 4$, Lamm [10] established the existence of global smooth solutions to (1.5)-(1.7) for $u_0 \in W^{2,2}(\Omega, N)$ with small $W^{2,2}$ -norm, and Gastel [4] and Wang [23] independently showed that there exists a unique global weak solution to (1.5)-(1.7) for any initial data $u_0 \in W^{2,2}(\Omega, N)$ that has at most finitely many singular times. Moreover, such a solution enjoys the energy inequality:

$$2 \int_0^T \int_{\Omega} |u_t|^2 + \int_{\Omega} |\Delta u|^2(T) \leq \int_{\Omega} |\Delta u_0|^2, \quad \forall 0 < T < +\infty. \quad (1.8)$$

Recently, Moser [15] was able to show the existence of a global weak solution to (1.5)-(1.7) for any target manifold N in dimensions $n \leq 8$.

It follows from (1.8) that there exists a sequence $t_m \uparrow \infty$ such that $u_m := u(\cdot, t_m) \in W^{2,2}(\Omega, N)$ satisfies

- (i) $\tau_2(u_m) := \|u_t(t_m)\|_{L^2} \rightarrow 0$; and
- (ii) u_m satisfies in the distribution sense

$$-\Delta^2 u_m + \Delta(\mathbb{B}(u_m)(\nabla u_m, \nabla u_m)) + 2\nabla \cdot \langle \Delta u_m, \nabla(\mathbb{P}(u_m)) \rangle - \langle \Delta(\mathbb{P}(u_m)), \Delta u_m \rangle = \tau_2(u_m). \quad (1.9)$$

In particular, when $N = \mathbb{S}^k$, by Definition 1.1 $\{u_m\}$ is a sequence of approximate biharmonic maps to \mathbb{S}^k , which are bounded in $W^{2,2}$ and have their bi-tension fields bounded in L^2 .

As an immediate corollary, we obtain the following theorem for the heat flow of biharmonic maps to \mathbb{S}^k in dimension 4.

Theorem 1.3 *For $n = 4$, $N = \mathbb{S}^k$, and $u_0 \in W^{2,2}(\Omega, \mathbb{S}^k)$, let $u : \Omega \times [0, +\infty) \rightarrow \mathbb{S}^k$, with $u \in L^\infty([0, +\infty), W^{2,2}(\Omega))$ and $u_t \in L^2([0, +\infty), L^2(\Omega))$, be a global weak solution of (1.5)-(1.7) that satisfies the energy inequality (1.8). Then there exist $t_m \uparrow +\infty$, a biharmonic map $u_\infty \in C^\infty \cap W^{2,2}(\Omega, \mathbb{S}^k)$ with $u_\infty = u_0$ on $\partial\Omega$, and a nonnegative integer L and $\{x_1, \dots, x_L\} \subset \Omega$ such that*

- (i) $u_m := u(\cdot, t_m) \rightharpoonup u_\infty$ in $W^{2,2}(\Omega, \mathbb{S}^k)$.
- (ii) $u_m \rightarrow u_\infty$ in $C_{\text{loc}}^0 \cap W_{\text{loc}}^{2,2}(\Omega \setminus \{x_1, \dots, x_L\}, \mathbb{S}^k)$.

(iii) For $1 \leq i \leq L$, there exist a positive integer L_i and nontrivial biharmonic maps $\{\omega_{ij}\}_{j=1}^{L_i}$ on \mathbb{R}^4 with finite Hessian energies such that

$$\lim_{m \rightarrow \infty} \int_{B_{r_i}(x_i)} |\nabla^2 u_m|^2 = \int_{B_{r_i}(x_i)} |\nabla^2 u_\infty|^2 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla^2 \omega_{ij}|^2, \quad (1.10)$$

and

$$\lim_{m \rightarrow \infty} \int_{B_{r_i}(x_i)} |\nabla u_m|^4 = \int_{B_{r_i}(x_i)} |\nabla u_\infty|^4 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla \omega_{ij}|^4, \quad (1.11)$$

where $r_i = \frac{1}{2} \min_{1 \leq j \leq L, j \neq i} \{|x_i - x_j|, \text{dist}(x_i, \partial\Omega)\}$.

In a forthcoming article [24], we will show Theorem 1.3 remains to be true for a general target manifold N .

The paper is organized as follows. In §2, we establish the Hölder continuity for any approximate biharmonic map with its bi-tension field in L^p for some $p > 1$, and $L^{2,1}$ -estimate of its Hessian $\nabla^2 u$. In §3, we show the strong convergence under the smallness condition of Hessian energy and set up the bubbling process. In §4, we show no concentration of $\|\nabla^2 u\|_{L^{2,\infty}(\cdot)}$ in the neck region. In §5, we apply the duality between $L^{2,1}$ and $L^{2,\infty}$ to show neither Hessian energy nor oscillation can concentrate in the neck region, which proves Theorem 1.2.

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2 A priori estimates of approximate biharmonic maps

This section is devoted to the estimate of $L^{2,1}$ norm of $\nabla^2 u$ for an approximate biharmonic map in terms of its Hessian energy and L^p -norm of its bi-tension field for some $p > 1$, and the Hölder continuity estimate under the smallness condition on its Hessian energy.

First we recall the definition and some basic properties of Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$ on \mathbb{R}^n (see [6] for more detail).

Definition 2.1 Let $U \subset \mathbb{R}^n$ be an open subset. For $1 < p < +\infty$ and $1 \leq q \leq \infty$, the Lorentz space $L^{p,q}(U)$ consists of all measurable functions $f : U \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^{p,q}(U)} = \begin{cases} \left(\int_0^{+\infty} (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < +\infty \\ \left\| t^{\frac{1}{p}} f^*(t) \right\|_{L^\infty(0,+\infty)}, & \text{if } q = +\infty \end{cases}$$

is finite, where $f^* : [0, |U|) \rightarrow \mathbb{R}$ denotes the nonincreasing rearrangement of $|f|$:

$$\left| \left\{ x \in U : |f(x)| \geq s \right\} \right| = \left| \left\{ t \in [0, |U|) : f^*(t) \geq s \right\} \right|, \quad \forall s \geq 0.$$

It is well-known that for $1 < p < +\infty$ and $1 \leq q \leq +\infty$, $L^{p,q}(U)$ is the dual space of $L^{\frac{p}{p-1}, \frac{q}{q-1}}(U)$. Moreover, $L^{p,p}(U) = L^p(U)$, $L^{p',q'}(U) \subset L^{p,q}(U)$ if $1 < p \leq p' < +\infty$ and $1 \leq q' \leq q \leq +\infty$ and $|U| < +\infty$, and $f \in L^{p,\infty}(U)$ is equivalent to

$$\|f\|_{L^{p,\infty}(U)} := \sup_{t>0} t \left| \{x \in U : |f(x)| \geq t\} \right|^{\frac{1}{p}} < +\infty. \quad (2.1)$$

We also recall the Sobolev embedding inequality between Lorentz spaces, whose proof can be found in [6].

Proposition 2.2 *For $1 \leq p < n$ and $1 \leq q < +\infty$, if $f \in L^p(\mathbb{R}^n)$ and its distributional derivative $\nabla f \in L^{p,q}(\mathbb{R}^n)$, then $f \in L^{\frac{np}{n-p}, q}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{\frac{np}{n-p}, q}(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^{p,q}(\mathbb{R}^n)}. \quad (2.2)$$

Now we have

Theorem 2.3 *For $n = 4$, suppose $u \in W^{2,2}(\Omega, \mathbb{S}^k)$ is an approximate biharmonic map with its bi-tension field $h \in L^p(\Omega)$ for some $1 < p < 2$. Then for any ball $B_\delta \subset \Omega$ with radius $\delta > 0$, $\nabla^2 u \in L^{2,1}(B_{\frac{\delta}{2}})$. Moreover, for any $0 < \theta \leq \frac{1}{2}$ the following estimates hold*

$$\|\nabla^2 u\|_{L^{2,1}(B_{\theta\delta})} \leq C \left(\theta \|\nabla^2 u\|_{L^2(B_\delta)} + \|\nabla^2 u\|_{L^2(B_\delta)}^2 + \delta^{4(1-\frac{1}{p})} \|h\|_{L^p(B_\delta)} \right), \quad (2.3)$$

and

$$\|\nabla^2 u\|_{L^2(B_{\theta\delta})} \leq C \left(\theta \|\nabla^2 u\|_{L^2(B_\delta)} + \|\nabla^2 u\|_{L^2(B_\delta)}^2 + \delta^{4(1-\frac{1}{p})} \|h\|_{L^p(B_\delta)} \right). \quad (2.4)$$

Proof. Since

$$\|\nabla^2 u\|_{L^2(B_{\theta\delta})} \leq \|\nabla^2 u\|_{L^{2,1}(B_{\theta\delta})},$$

(2.4) follows directly from (2.3). The idea to prove (2.3) is similar to that of [20]. Let \times denote the wedge product in \mathbb{R}^{k+1} . First observe that the equation (1.1) is equivalent to:

$$\Delta(\nabla \cdot (\nabla u \times u)) = 2\nabla \cdot (\Delta u \times \nabla u) + h \times u. \quad (2.5)$$

Since by scaling the case $\delta \neq 1$ can be reduced to the case $\delta = 1$, we assume $\delta = 1$ for simplicity. Let $\tilde{h} \in L^p(\mathbb{R}^4)$ be an extension of h such that

$$\|\tilde{h}\|_{L^q(\mathbb{R}^4)} \leq C \|h\|_{L^q(B_1)}, \quad \forall 1 \leq q \leq p. \quad (2.6)$$

Let $\tilde{u} \in W^{2,2}(\mathbb{R}^4, \mathbb{R}^{k+1})$ be an extension of u such that

$$\|\nabla \tilde{u}\|_{L^4(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^4(B_1)}, \quad \|\nabla^2 \tilde{u}\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla^2 u\|_{L^2(B_1)}. \quad (2.7)$$

Now we consider the Hodge decomposition of the 1-form $d\tilde{u} \times \tilde{u} = \sum_{i=1}^4 \frac{\partial \tilde{u}}{\partial x_i} \times \tilde{u} dx_i \in L^4(\mathbb{R}^4, \wedge^1 \mathbb{R}^4)$.

It is well-known [8] that there exist a function $F \in W^{1,4}(\mathbb{R}^4)$ and a 2-form $H \in W^{1,4}(\mathbb{R}^4, \wedge^2 \mathbb{R}^4)$ such that

$$d\tilde{u} \times \tilde{u} = dF + d^* H, \quad dH = 0 \quad \text{in } \mathbb{R}^4, \quad (2.8)$$

$$\|\nabla F\|_{L^4(\mathbb{R}^4)} + \|\nabla H\|_{L^4(\mathbb{R}^4)} \leq C \|\nabla u\|_{L^4(B_1)}, \quad (2.9)$$

and

$$\|\nabla^2 F\|_{L^2(\mathbb{R}^4)} + \|\nabla^2 H\|_{L^2(\mathbb{R}^4)} \leq C \|\nabla^2 u\|_{L^2(B_1)}, \quad (2.10)$$

It is easy to see that H satisfies

$$\Delta H = d\tilde{u} \times d\tilde{u} \text{ in } \mathbb{R}^4. \quad (2.11)$$

By Proposition 2.2 and Hölder inequality, $d\tilde{u} \times d\tilde{u} \in L^{2,1}(\mathbb{R}^4)$. Hence, by the Calderon-Zygmund's $L^{p,q}$ -theory, we conclude that $\nabla^2 H \in L^{2,1}(\mathbb{R}^4)$, and

$$\begin{aligned} \|\nabla^2 H\|_{L^{2,1}(\mathbb{R}^4)} &\leq C \|d\tilde{u} \times d\tilde{u}\|_{L^{2,1}(\mathbb{R}^4)} \leq C \|\nabla \tilde{u}\|_{L^{4,2}(\mathbb{R}^4)}^2 \\ &\leq C \|\nabla^2 \tilde{u}\|_{L^{2,2}(\mathbb{R}^4)}^2 \leq C \|\nabla^2 u\|_{L^2(B_1)}^2. \end{aligned} \quad (2.12)$$

To estimate $\|\nabla^2 F\|_{L^{2,1}}$, let $G(x-y) = c_4 \ln|x-y|$ be the fundamental solution of Δ^2 on \mathbb{R}^4 . Set $F_1, F_2 : \mathbb{R}^4 \rightarrow \mathbb{R}^{k+1}$ by letting

$$F_1(x) = -2 \int_{\mathbb{R}^4} \nabla_y G(x-y) \cdot (\Delta \tilde{u} \times \nabla \tilde{u})(y) dy, \quad (2.13)$$

and

$$F_2(x) = \int_{\mathbb{R}^4} G(x-y) (\tilde{h} \times \tilde{u})(y) dy. \quad (2.14)$$

Then it is readily seen that

$$\Delta^2 F_1 = 2\nabla \cdot (\Delta \tilde{u} \times \nabla \tilde{u}), \text{ in } \mathbb{R}^4, \quad (2.15)$$

and

$$\Delta^2 F_2 = \tilde{h} \times \tilde{u}, \text{ in } \mathbb{R}^4. \quad (2.16)$$

Since F satisfies

$$\Delta^2 F = 2\nabla \cdot (\Delta u \times \nabla u) + h \times u, \text{ in } B_1, \quad (2.17)$$

we conclude that $F_3 := F - F_1 - F_2$ satisfies

$$\Delta^2 F_3 = 0, \text{ in } B_1. \quad (2.18)$$

Now we want to estimate F_1, F_2, F_3 as follows. For F_1 , since

$$\nabla^3 F_1(x) = -2 \int_{\mathbb{R}^4} \nabla_y^4 G(x-y) \cdot (\Delta \tilde{u} \times \nabla \tilde{u})(y) dy, \quad (2.19)$$

and by Hölder inequality $\Delta \tilde{u} \times \nabla \tilde{u} \in L^{\frac{4}{3},1}(\mathbb{R}^4)$, we have by Calderon-Zygmund $L^{p,q}$ -theory that $\nabla^3 F_1 \in L^{\frac{4}{3},1}(\mathbb{R}^4)$ and

$$\|\nabla^3 F_1\|_{L^{\frac{4}{3},1}(\mathbb{R}^4)} \leq C \|\Delta \tilde{u} \times \nabla \tilde{u}\|_{L^{\frac{4}{3},1}(\mathbb{R}^4)} \leq C \|\nabla^2 \tilde{u}\|_{L^2(\mathbb{R}^4)} \|\nabla \tilde{u}\|_{L^{4,2}(\mathbb{R}^4)} \leq C \|\nabla^2 u\|_{L^2(B_1)}^2. \quad (2.20)$$

Hence, by Proposition 2.2 we have that $\nabla^2 F_1 \in L^{2,1}(\mathbb{R}^4)$ and

$$\|\nabla^2 F_1\|_{L^{2,1}(\mathbb{R}^4)} \leq C \|\nabla^3 F_1\|_{L^{\frac{4}{3},1}(\mathbb{R}^4)} \leq C \|\nabla^2 u\|_{L^2(B_1)}^2. \quad (2.21)$$

For F_2 , we have

$$|\nabla^2 F_2(x)| \leq c_4 \left| \int_{\mathbb{R}^4} |x-y|^{-2} (\tilde{h} \times \tilde{u})(y) dy \right| \leq C I_2(|\tilde{h}|)(x)$$

where $I_\beta(f)$ is the Riesz potential of order β ($0 < \beta \leq 4$) defined by

$$I_\beta(f)(x) \equiv \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^{4-\beta}} dy, \quad x \in \mathbb{R}^4. \quad (2.22)$$

It follows from Adams [1] (see also [20]) that $\nabla^2 F_2 \in L^{\frac{2p}{2-p}}(\mathbb{R}^4)$ and

$$\left\| \nabla^2 F_2 \right\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^4)} \leq C \left\| I_2(\tilde{h}) \right\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^4)} \leq C \left\| \tilde{h} \right\|_{L^p(\mathbb{R}^4)} \leq C \left\| h \right\|_{L^p(B_1)}. \quad (2.23)$$

Since $L^{\frac{2p}{2-p}}(B_1) \subset L^{2,1}(B_1)$, this implies

$$\left\| \nabla^2 F_2 \right\|_{L^{2,1}(B_1)} \leq C \left\| \nabla^2 F_2 \right\|_{L^{\frac{2p}{2-p}}(B_1)} \leq C \left\| h \right\|_{L^p(B_1)}. \quad (2.24)$$

Since F_3 is a biharmonic function on B_1 , the standard estimate implies that for any $0 < \theta \leq \frac{1}{2}$,

$$\begin{aligned} \left\| \nabla^2 F_3 \right\|_{L^{2,1}(B_\theta)} &\leq C\theta \left\| \nabla^2 F_3 \right\|_{L^2(B_1)} \\ &\leq C\theta \left(\left\| \nabla^2 F \right\|_{L^2(B_1)} + \left\| \nabla^2 F_1 \right\|_{L^2(B_1)} + \left\| \nabla^2 F_2 \right\|_{L^2(B_1)} \right) \\ &\leq C\theta \left(\left\| \nabla^2 u \right\|_{L^2(B_1)} + \left\| h \right\|_{L^p(B_1)} \right). \end{aligned} \quad (2.25)$$

Combining (2.21), (2.23) together with (2.25) yields

$$\left\| \nabla^2 F \right\|_{L^{2,1}(B_\theta)} \leq C \left(\theta \left\| \nabla^2 u \right\|_{L^2(B_1)} + \left\| h \right\|_{L^p(B_1)} \right). \quad (2.26)$$

Since $|u| = 1$, we have

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \cdot u + \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} = 0, \quad (2.27)$$

and hence

$$\begin{aligned} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| &\leq \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot u \right| + \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \times u \right| \\ &\leq 2 \left| \frac{\partial u}{\partial x_i} \right| \left| \frac{\partial u}{\partial x_j} \right| + \left| \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_j} \times u \right) \right| \end{aligned} \quad (2.28)$$

$$\leq 2|\nabla u|^2 + C(|\nabla^2 F| + |\nabla^2 H|), \quad (2.29)$$

and by (2.12) and (2.26) we have

$$\begin{aligned} \left\| \nabla^2 u \right\|_{L^{2,1}(B_\theta)} &\leq C \left(\left\| |\nabla u|^2 \right\|_{L^{2,1}(B_\theta)} + \left\| \nabla^2 F \right\|_{L^{2,1}(B_\theta)} + \left\| \nabla^2 H \right\|_{L^{2,1}(B_\theta)} \right) \\ &\leq C \left(\theta \left\| \nabla^2 u \right\|_{L^2(B_1)} + \left\| \nabla^2 u \right\|_{L^2(B_1)}^2 + \left\| h \right\|_{L^p(B_1)} \right). \end{aligned}$$

This clearly yields (2.3). \square

As a corollary of Theorem 2.3 and (2.4), we can show that an approximate biharmonic map to \mathbb{S}^k , with the bi-tension field in L^p for some $1 < p < 2$, is Hölder continuous. More precisely, we have

Corollary 2.4 For $n = 4$, there exist an $\epsilon_0 > 0$, $\alpha \in (0, 1)$, and $2 < q \leq \frac{2p}{2-p}$ such that if $u \in W^{2,2}(\Omega, \mathbb{S}^k)$ is an approximate biharmonic map with its bi-tension field $h \in L^p(\Omega)$ for some $1 < p < 2$, that satisfies

$$\left\| \nabla^2 u \right\|_{L^2(B_r(x))} \leq \epsilon_0, \quad (2.30)$$

for some $B_r(x) \subset \Omega$, then

(i) $u \in C^\alpha(B_r(x), \mathbb{S}^k)$ and

$$\left[u \right]_{C^\alpha(B_{\frac{r}{2}}(x))} \leq C \left(\epsilon_0, \|h\|_{L^p(B_r(x))} \right), \quad (2.31)$$

(ii) $\nabla^2 u \in L^q(B_{\frac{r}{2}})$ and

$$\left\| \nabla^2 u \right\|_{L^q(B_{\frac{r}{2}})} \leq C \left(\epsilon_0, \|h\|_{L^p(B_r(x))} \right). \quad (2.32)$$

Proof. For simplicity, assume $x = 0$ and $r = 1$. For any ball $B_\delta(y) \subset B_1 \subset \Omega$, (2.30) yields

$$\left\| \nabla^2 u \right\|_{L^2(B_\delta(y))} \leq \epsilon_0$$

so that applying (2.4), we have that for any $\theta \in (0, \frac{1}{2})$,

$$\left\| \nabla^2 u \right\|_{L^2(B_{\theta\delta}(y))} \leq C(\theta + \epsilon_0) \left\| \nabla^2 u \right\|_{L^2(B_\delta(y))} + C\delta^{4(1-\frac{1}{p})} \|h\|_{L^p(B_\delta(y))}. \quad (2.33)$$

Thus by choosing sufficiently small $\epsilon_0 > 0$ and $\theta = \theta_0 \in (0, \frac{1}{2})$ we have

$$\left\| \nabla^2 u \right\|_{L^2(B_{\theta_0\delta}(y))} \leq \frac{1}{2} \left\| \nabla^2 u \right\|_{L^2(B_\delta(y))} + C\delta^{4(1-\frac{1}{p})} \|h\|_{L^p(B_\delta(y))}. \quad (2.34)$$

It is standard that iterations of (2.33) would imply that there exists $\alpha \in (0, 1)$ such that for any $y \in B_{\frac{1}{2}}$ and $0 < R \leq \frac{1}{4}$,

$$\int_{B_R(y)} |\nabla^2 u|^2 \leq CR^{2\alpha} \left(\int_{B_1} |\nabla^2 u|^2 + \|h\|_{L^p(B_1)}^2 \right). \quad (2.35)$$

This, combined with Morrey's decay Lemma, clearly implies that $u \in C^\alpha(B_{\frac{1}{2}})$ and

$$\left[u \right]_{C^\alpha(B_{\frac{1}{2}})} \leq C \left(\left\| \nabla^2 u \right\|_{L^2(B_1)} + \|h\|_{L^p(B_1)} \right).$$

This proves (i).

(ii) can be proved by applying Adams's Riesz potential estimate between Morrey spaces. First recall that for an open set $U \subset \mathbb{R}^4$, $1 \leq p < +\infty$, $0 < \lambda \leq 4$, the Morrey space $M^{p,\lambda}(U)$ is defined by

$$M^{p,\lambda}(U) = \left\{ f \in L^p(U) : \|f\|_{M^{p,\lambda}}^p = \sup_{B_r \subset U} r^{\lambda-4} \int_{B_r} |f|^p < +\infty \right\}.$$

By (i), we know that for some $0 < \alpha < 1$, $\nabla^2 u \in M^{2,4-2\alpha}(B_{\frac{3}{4}})$ and

$$\left\| \nabla^2 u \right\|_{M^{2,4-2\alpha}(B_{\frac{3}{4}})} \leq C \left(\left\| \nabla^2 u \right\|_{L^2(B_1)} + \|h\|_{L^p(B_1)} \right). \quad (2.36)$$

Let $\tilde{u} : \mathbb{R}^4 \rightarrow \mathbb{R}^{k+1}$ be an extension of u on $B_{\frac{3}{4}}$ such that

$$\left\| \nabla^2 \tilde{u} \right\|_{M^{2,4-2\alpha}(\mathbb{R}^4)} \leq C \left\| \nabla^2 u \right\|_{M^{2,4-2\alpha}(B_{\frac{3}{4}})}. \quad (2.37)$$

Then there exists a harmonic function $v : B_{\frac{3}{4}} \rightarrow \mathbb{R}^{k+1}$, with $\|\nabla^2 v\|_{L^2(B_{\frac{3}{4}})} \leq C \|\nabla^2 u\|_{L^2(B_1)}$, such that $u = v + w$ on $B_{\frac{3}{4}}$, where

$$w(x) = \int_{\mathbb{R}^4} \frac{\Delta \tilde{u}(y)}{|x-y|^2} dy.$$

Hence

$$|\nabla w|(x) \leq C I_1(|\Delta \tilde{u}|)(x), \quad x \in \mathbb{R}^4.$$

By Adams's Riesz potential estimate (see [1] or [20]), we have $|\nabla w| \in M^{\frac{4-2\alpha}{1-\alpha}, 4-2\alpha}(\mathbb{R}^4)$. Therefore we can conclude that $\nabla u \in M^{\frac{4-2\alpha}{1-\alpha}, 4-2\alpha}(B_{\frac{5}{8}})$ and

$$\left\| \nabla u \right\|_{L^{\frac{4-2\alpha}{1-\alpha}}(B_{\frac{5}{8}})} \leq \left\| \nabla u \right\|_{M^{\frac{4-2\alpha}{1-\alpha}, 4-2\alpha}(\mathbb{R}^4)} \leq C \left(\|\nabla^2 u\|_{L^2(B_1)} + \|h\|_{L^p(B_1)} \right). \quad (2.38)$$

Denote $q_1 = \frac{4-2\alpha}{1-\alpha} > 4$. Then we have $\Delta u \times \nabla u \in L^{\frac{2q_1}{2+q_1}}(B_{\frac{5}{8}})$. Note that $q_2 = \frac{2q_1}{2+q_1} > \frac{4}{3}$. Let $C : B_{\frac{5}{8}} \rightarrow \mathbb{R}^{k+1}$ be the harmonic function extension of $\nabla \cdot (\nabla u \times u)$ from $\partial B_{\frac{5}{8}}$ to $B_{\frac{5}{8}}$. Then, by the equation (2.5), we have that in $B_{\frac{5}{8}}$,

$$\begin{aligned} \nabla \cdot (\nabla u \times u)(x) &= -2 \int_{B_{\frac{5}{8}}} \nabla_y \hat{G}(x-y) (\Delta u \times \nabla u)(y) dy + \int_{B_{\frac{5}{8}}} \hat{G}(x-y) (h \times u)(y) dy + C(x) \\ &= A(x) + B(x) + C(x), \end{aligned}$$

where \hat{G} is the Green function of Δ on $B_{\frac{5}{8}}$. It is easy to see that

$$|A(x)| \leq C |I_1(x)| \leq I_1(|\Delta u| |\nabla u| \chi_{B_{\frac{5}{8}}})(x), \quad x \in B_{\frac{5}{8}},$$

where $\chi_{B_{\frac{5}{8}}}$ is the characteristic function of $B_{\frac{5}{8}}$. Hence we have $A(x) \in L^{\frac{4q_2}{4-q_2}}(B_{\frac{5}{8}})$, and

$$\left\| A(x) \right\|_{L^{\frac{4q_2}{4-q_2}}(B_{\frac{5}{8}})} \leq C \left\| |\Delta u| |\nabla u| \right\|_{L^{q_2}(B_{\frac{5}{8}})} \leq C \left(\|\nabla^2 u\|_{L^2(B_1)} + \|h\|_{L^p(B_1)} \right).$$

For $B(x)$, it is easy to see that

$$|B(x)| \leq C I_2(|h|)(x), \quad \forall x \in B_{\frac{5}{8}}.$$

Hence $B(x) \in L^{\frac{2p}{2-p}}(B_{\frac{5}{8}})$, and

$$\left\| B(x) \right\|_{L^{\frac{2p}{2-p}}(B_{\frac{5}{8}})} \leq C \left\| h \right\|_{L^p(B_1)}.$$

It is easy to see that

$$\left\| C(x) \right\|_{L^\infty(B_{\frac{1}{2}})} \leq \left\| \nabla \cdot (\nabla u \times u) \right\|_{L^2(\partial B_{\frac{5}{8}})} \leq C \left\| \nabla^2 u \right\|_{L^2(B_1)}.$$

Set $q = \min\{\frac{2p}{2-p}, \frac{4q_2}{4-q_2}\}$. Since $q_2 > \frac{4}{3}$, we have $q > 2$ and

$$\left\| \nabla \cdot (\nabla u \times u) \right\|_{L^q(B_{\frac{1}{2}})} \leq C \left(\|\nabla^2 u\|_{L^2(B_1)} + \|h\|_{L^p(B_1)} \right).$$

This, combined with (2.28) and (2.38), implies that (2.32) holds for some $q > 2$. The proof is now complete. \square

3 Blow up analysis and energy inequality

This section is devoted to ϵ_0 -compactness lemma and preliminary steps on the blow up analysis of approximate biharmonic maps with bi-tension fields bounded in L^p for some $p > 1$. In particular, we will indicate that (1.10) holds with “=” replaced by “ \geq ”.

First we have

Lemma 3.1 *For $n = 4$, there exists an $\epsilon_0 > 0$ such that if $\{u_m\} \subset W^{2,2}(B_1, \mathbb{S}^k)$ is a sequence of approximate biharmonic maps satisfying*

$$\sup_m \|\nabla^2 u_m\|_{L^2(B_1)} \leq \epsilon_0, \quad (3.1)$$

and $u_m \rightharpoonup u$ in $W^{2,2}(B_1)$ and $h_m \rightharpoonup h$ in $L^p(B_1)$ for some $p > 1$. Then u is an approximate biharmonic map with bi-tension field h , and

$$\lim_{m \rightarrow \infty} \|u_m - u\|_{W^{2,2}(B_{\frac{1}{2}})} = 0. \quad (3.2)$$

Proof. The first assertion follows easily from (2.5). To show (3.2), it suffices to show that $\{u_m\}$ is a Cauchy sequence in $W^{2,2}(B_{\frac{1}{2}})$. By (3.1) and Corollary 2.4, there exist $\alpha \in (0, 1)$ and $q > 2$ such that

$$\sup_m \left[\|u_m\|_{C^\alpha(B_{\frac{3}{4}})} + \|\nabla^2 u_m\|_{L^q(B_{\frac{3}{4}})} \right] \leq C.$$

Hence we may assume that

$$\lim_{m, l \rightarrow \infty} \|u_m - u_l\|_{L^\infty(B_{\frac{3}{4}})} = 0.$$

For $\eta \in C_0^\infty(B_{\frac{3}{4}})$ be a cut-off function of $B_{\frac{1}{2}}$, multiplying the equations of u_m and u_l by $(u_m - u_l)\phi^2$ and integrating over B_1 , we obtain

$$\begin{aligned} & \int_{B_1} |\Delta(u_m - u_l)|^2 \phi^2 \\ & \leq \int_{B_1} |\Delta(u_m - u_l)| (2|\nabla(u_m - u_l)| |\nabla \phi^2| + |u_m - u_l| |\Delta \phi^2|) + \int_{B_1} |h_m - h_l| |u_m - u_l| \phi^2 \\ & \quad + 3 \int_{B_1} (|\Delta u_l|^2 + |\Delta u_m|^2) |u_m - u_l| \phi^2 \\ & \quad + 4 \int_{B_1} |\nabla^2 u_m| |\nabla u_m| |\nabla(u_m(u_m - u_l)\phi^2)| \\ & \quad + 4 \int_{B_1} |\nabla^2 u_l| |\nabla u_l| |\nabla(u_l(u_m - u_l)\phi^2)| \\ & = I + II + III + IV + V. \end{aligned}$$

It is easy to see

$$|I| \leq C(\|\nabla(u_m - u_l)\|_{L^2(B_{\frac{3}{4}})} + \|u_m - u_l\|_{L^\infty(B_{\frac{3}{4}})}) \rightarrow 0,$$

$$|II| \leq C\|h_m - h_l\|_{L^1(B_{\frac{3}{4}})}\|u_m - u_l\|_{L^\infty(B_{\frac{3}{4}})} \rightarrow 0,$$

$$|III| \leq C(\|\nabla^2 u_m\|_{L^2(B_{\frac{3}{4}})}^2 + \|\nabla^2 u_l\|_{L^2(B_{\frac{3}{4}})}^2)\|u_m - u_l\|_{L^\infty(B_{\frac{3}{4}})} \rightarrow 0.$$

For IV , observe that for $1 < r < 4$ with $\frac{1}{4} + \frac{1}{q} + \frac{1}{r} = 1$, we have

$$\begin{aligned} |IV| \leq & C\left(\|\nabla^2 u_m\|_{L^2(B_{\frac{3}{4}})}\|\nabla u_l\|_{L^4(B_{\frac{3}{4}})}^2\|u_m - u_l\|_{L^\infty(B_{\frac{3}{4}})}\right. \\ & \left. + \|\nabla^2 u_m\|_{L^q(B_{\frac{3}{4}})}\|\nabla u_m\|_{L^4(B_{\frac{3}{4}})}\|\nabla(u_m - u_l)\|_{L^r(B_{\frac{3}{4}})}\right) \rightarrow 0, \end{aligned}$$

since $\|\nabla(u_m - u_l)\|_{L^r(B_{\frac{3}{4}})} \rightarrow 0$. Similarly, we can show

$$|V| \rightarrow 0.$$

Hence $\{u_m\}$ is a Cauchy sequence in $W^{2,2}(B_{\frac{1}{2}})$. This completes the proof. \square

Lemma 3.2 *Under the same assumptions as Theorem 1.2, there exists a finite subset $\Sigma \subset \Omega$ such that $u_m \rightarrow u$ in $W_{\text{loc}}^{2,2} \cap C_{\text{loc}}^0(\Omega \setminus \Sigma, \mathbb{S}^k)$. Moreover, $u \in W^{2,2} \cap C^0(\Omega, \mathbb{S}^k)$ is an approximate biharmonic map with bi-tension field h .*

Proof. Let $\epsilon_0 > 0$ be given by Corollary 2.4, and define

$$\Sigma := \bigcap_{r>0} \left\{ x \in \Omega : \liminf_{m \rightarrow \infty} \int_{B_r(x)} |\nabla^2 u_m|^2 > \epsilon_0^2 \right\}. \quad (3.3)$$

Then by a simple covering argument we have that Σ is a finite set. In fact

$$H^0(\Sigma) \leq \frac{1}{\epsilon_0^2} \sup_m \int_{\Omega} |\nabla^2 u_m|^2 < +\infty.$$

For any $x_0 \in \Omega \setminus \Sigma$, there exists $r_0 > 0$ such that

$$\liminf_{m \rightarrow \infty} \int_{B_{r_0}(x_0)} |\nabla^2 u_m|^2 \leq \epsilon_0^2.$$

Hence Corollary 2.4 and Lemma 3.1 imply that there exists $\alpha \in (0, 1)$ such that

$$\|u_m\|_{C^\alpha(B_{\frac{r_0}{2}}(x_0))} \leq C,$$

so that $u_m \rightarrow u$ in $C^0 \cap W^{2,2}(B_{\frac{r_0}{2}}(x_0))$. This proves that $u_m \rightarrow u$ in $W_{\text{loc}}^{2,2} \cap C_{\text{loc}}^0(\Omega \setminus \Sigma)$. It is clear that $u \in W^{2,2}(\Omega)$ is an approximate biharmonic map with bi-tension field $h \in L^p(\Omega)$. Applying Corollary 2.4 again, we conclude that $u \in C^0(\Omega, \mathbb{S}^k)$. \square

Proof of Theorem 1.2:

The proof of (1.10) with “ \geq ” is similar to [20] Lemma 3.3. Here we sketch it. For any $x_0 \in \Sigma$, there exist $r_0 > 0$, $x_m \rightarrow x_0$ and $r_m \downarrow 0$ such that

$$\max_{x \in B_{r_0}(x_0)} \left\{ \int_{B_{r_m}(x)} |\nabla^2 u_m|^2 \right\} = \frac{\epsilon_0^2}{2} = \int_{B_{r_m}(x_m)} |\nabla^2 u_m|^2.$$

Define $v_m(x) = u_m(x_m + r_m x) : B_{\frac{r_0}{r_m}} \rightarrow \mathbb{S}^k$. Then v_m is an approximate biharmonic map, with bi-tension field $\widetilde{h}_m(x) = r_m^4 h(x_m + r_m x)$, that satisfies

$$\int_{B_1(x)} |\nabla^2 u_m|^2 \leq \frac{\epsilon_0^2}{2}, \quad \forall x \in B_{\frac{r_0}{r_m}}, \quad \text{and} \quad \int_{B_1} |\nabla^2 u_m|^2 = \frac{\epsilon_0^2}{2},$$

and

$$\left\| \widetilde{h}_m \right\|_{L^p(B_{\frac{r_0}{r_m}})} \leq r_m^{4(1-\frac{1}{p})} \left\| h_m \right\|_{L^p(\Omega)} \rightarrow 0.$$

Thus Corollary 2.4 and Lemma 3.1 imply that, after taking possible subsequences, there exists a nontrivial biharmonic map $\omega : \mathbb{R}^4 \rightarrow \mathbb{S}^k$ with

$$\frac{\epsilon_0^2}{2} \leq \int_{\mathbb{R}^4} |\nabla^2 \omega|^2 < +\infty$$

such that $v_m \rightarrow \omega$ in $W_{\text{loc}}^{2,2} \cap C_{\text{loc}}^0(\mathbb{R}^4)$. Performing such a blow-up argument near any $x_i \in \Sigma$, $1 \leq i \leq L$, we can find all possible nontrivial biharmonic maps $\{\omega_{ij}\} \in W^{2,2}(\mathbb{R}^4)$ for $1 \leq j \leq L_i$, with $L_i \leq CM\epsilon_0^{-2}$. It is not hard to see (1.10) holds with “=” replaced by “ \geq ”.

In order to prove “ \leq ” of (1.10), we need to show that the $L^{2,\infty}$ -norm of u_m over any neck region is arbitrarily small. This will be done in the next section. We will return to the proof of Theorem 1.2 in §5.

4 $L^{2,\infty}$ -estimate in the neck region

In this section, we first show that there is no concentration of $\|\nabla^2 u\|_{L^{2,\infty}}$ in the neck region. Then use the duality between $L^{2,1}$ and $L^{2,\infty}$ to prove Theorem 1.2 by showing that there is no Hessian energy concentration in the neck region. More precisely, we have

Lemma 4.1 *For any $\epsilon > 0$, suppose that $u \in W^{2,2}(B_1, \mathbb{S}^k)$ is an approximate biharmonic map, whose bi-tension field $h \in L^p(B_1)$ for some $p > 1$, satisfying that for $0 < \delta < \frac{1}{2}$, $R > 1$, and $0 < r < \frac{2\delta}{R}$,*

$$\int_{B_{2\rho} \setminus B_\rho} |\nabla^2 u|^2 dx \leq \epsilon^2, \quad \forall Rr \leq \rho \leq 2\delta, \quad (4.1)$$

then

$$\begin{aligned} \left\| \nabla^2 u \right\|_{L^{(2,\infty)}(B_\delta \setminus B_{2Rr})} &\leq C \left[\epsilon + \epsilon^{\frac{1}{2}} \|\nabla^2 u\|_{L^2(B_{2\delta})} + \delta^2 + \delta^{4(1-\frac{1}{p})} \left\| h \right\|_{L^p(B_{2\delta})} \right. \\ &\quad \left. + \epsilon^{-1} \delta^{8(1-\frac{1}{p})} \|h\|_{L^p(B_{2\delta})}^2 \right]. \end{aligned} \quad (4.2)$$

Proof. First recall that

$$\begin{aligned} \left\| \nabla^2 u \right\|_{L^{2,\infty}(B_\delta \setminus B_{2Rr})}^2 &= \sup_{\lambda > 0} \lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right| \\ &\leq \sup_{0 < \lambda \leq 1} \lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right| \\ &\quad + \sup_{\lambda > 1} \lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right| \\ &\leq C\delta^4 + \sup_{\lambda > 1} \lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right|. \end{aligned} \quad (4.3)$$

It suffices to estimate $\lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right|$ for $\lambda > 1$.

We may assume that $\delta = 2^K \sqrt{\frac{\epsilon}{\lambda}}$ for some positive integer $K \geq 1$. There are two cases to consider:

(i) $\sqrt{\frac{\epsilon}{\lambda}} \geq 2Rr$. Then we have

$$\begin{aligned} & \lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right| \\ & \leq \lambda^2 \left| \left\{ x \in B_{\sqrt{\frac{\epsilon}{\lambda}}} \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right| + \lambda^2 \left| \left\{ x \in B_\delta \setminus B_{\sqrt{\frac{\epsilon}{\lambda}}} : |\nabla^2 u| > \lambda \right\} \right| \\ & \leq C\epsilon^2 + \lambda^2 \sum_{i=0}^{K-1} \left| \left\{ x \in B_{2^{i+1}\sqrt{\frac{\epsilon}{\lambda}}} \setminus B_{2^i\sqrt{\frac{\epsilon}{\lambda}}} : |\nabla^2 u| > \lambda \right\} \right|. \end{aligned}$$

(ii) $\sqrt{\frac{\epsilon}{\lambda}} < 2Rr$. Then we may assume that there exists $1 \leq i_0 \leq K$ such that $2Rr = 2^{i_0} \sqrt{\frac{\epsilon}{\lambda}}$ so that

$$\lambda^2 \left| \left\{ x \in B_\delta \setminus B_{2Rr} : |\nabla^2 u| > \lambda \right\} \right| \leq \lambda^2 \sum_{i=i_0}^{K-1} \left| \left\{ x \in B_{2^{i+1}\sqrt{\frac{\epsilon}{\lambda}}} \setminus B_{2^i\sqrt{\frac{\epsilon}{\lambda}}} : |\nabla^2 u| > \lambda \right\} \right|.$$

It is not hard to see that the case (ii) can be done by the same way as the case (i). Thus we only need to prove (i). To simplify the presentation, introduce

$$r_0 := \sqrt{\frac{\epsilon}{\lambda}}, \mathcal{A}_i := B_{2^{i+1}r_0} \setminus B_{2^i r_0} \text{ and } \mathcal{B}_i := B_{2^{i+2}r_0} \setminus B_{2^{i-1}r_0}, \quad 0 \leq i \leq K-1.$$

For $0 \leq i \leq K-1$, let $u_i : \mathbb{R}^4 \rightarrow \mathbb{R}^{k+1}$ be an extension of u from \mathcal{B}_i to \mathbb{R}^4 such that

$$|u_i| \leq 1; \quad \int_{\mathbb{R}^4} (|\nabla^2 u_i|^2 + |\nabla u_i|^4) dx \leq C \int_{\mathcal{B}_i} (|\nabla^2 u|^2 + |\nabla u|^4) dx \leq C \int_{\mathcal{B}_i} |\nabla^2 u|^2 dx \leq C\epsilon^2, \quad (4.4)$$

where we have used $|\nabla u|^2 = |\Delta u \cdot u| \leq |\Delta u|$ on \mathcal{B}_i and (4.1). Let $h_i : \mathbb{R}^4 \rightarrow \mathbb{R}^{k+1}$ be an extension of h from \mathcal{B}_i to \mathbb{R}^4 such that

$$\|h_i\|_{L^q(\mathbb{R}^4)} \leq C \|h\|_{L^q(\mathcal{B}_i)}, \quad 1 \leq q \leq p. \quad (4.5)$$

Define $v_i : \mathbb{R}^4 \rightarrow \mathbb{R}^{k+1}$ by

$$v_i(x) = \int_{\mathbb{R}^4} G(x-y) [h_i - (|\Delta u_i|^2 + \Delta(|\nabla u_i|^2) + 2\langle \nabla u_i, \nabla \Delta u_i \rangle) u_i](y) dy, \quad (4.6)$$

where $G(\cdot)$ is the fundamental solution of Δ^2 on \mathbb{R}^4 , and $w_i : \mathcal{B}_i \rightarrow \mathbb{R}^{k+1}$ by

$$w_i = u - v_i. \quad (4.7)$$

Then it is easy to see that

$$\Delta^2 w_i = 0 \quad \text{on } \mathcal{B}_i. \quad (4.8)$$

Now we have

Claim. For $0 \leq i \leq K-1$,

$$\|\nabla^2 v_i\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C \left[\|h\|_{L^1(\mathcal{B}_i)} + \|\nabla^2 u\|_{L^2(\mathcal{B}_i)}^{\frac{3}{2}} \right]. \quad (4.9)$$

and

$$\sup_{x \in \mathcal{A}_i} \left\{ |x|^2 \left| \nabla^2 w_i \right| (x) \right\} \leq C \left[\|h\|_{L^1(\mathcal{B}_i)} + \|\nabla^2 u\|_{L^2(\mathcal{B}_i)} \right]. \quad (4.10)$$

To see (4.9), note that

$$\begin{aligned} \nabla^2 v_i(x) &= \int_{\mathbb{R}^4} \nabla_y^2 G(x-y) [h_i - (|\Delta u_i|^2 + \Delta(|\nabla u_i|^2) + 2\langle \nabla u_i, \nabla \Delta u_i \rangle) u_i] (y) dy \\ &= I(x) + II(x) + III(x) + IV(x). \end{aligned} \quad (4.11)$$

Since $|\nabla_y^2 G(x-y)| \leq \frac{C}{|x-y|^2} \in L^{2,\infty}(\mathbb{R}^4)$, we have

$$\|I\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C \|h_i\|_{L^1(\mathbb{R}^4)}$$

and

$$\|II\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C \left\| |\Delta u_i|^2 \right\|_{L^1(\mathbb{R}^4)} \leq C \left\| \nabla^2 u \right\|_{L^2(\mathbb{R}^4)}^2.$$

For III , by integration by parts we have

$$\begin{aligned} & \left| III(x) \right| \\ &= \left| \int_{\mathbb{R}^4} \nabla_y^2 G(x-y) \Delta(|\nabla u_i|^2) u_i(y) dy \right| \\ &\leq 2 \left(\int_{\mathbb{R}^4} |\nabla_y^3 G|(x-y) |\nabla^2 u_i(y)| |\nabla u_i(y)| dy + \int_{\mathbb{R}^4} |\nabla_y^2 G|(x-y) |\nabla^2 u_i(y)| |\nabla u_i(y)|^2 dy \right) \\ &\leq CI_1(|\nabla^2 u_i| |\nabla u_i|)(x) + CI_2(|\nabla^2 u_i| |\nabla u_i|^2)(x). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|III\|_{L^{2,\infty}(\mathbb{R}^4)} &\leq C \left(\left\| |\nabla^2 u_i| |\nabla u_i| \right\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} + \left\| |\nabla^2 u_i| |\nabla u_i|^2 \right\|_{L^1(\mathbb{R}^4)} \right) \\ &\leq C \left\| \nabla^2 u_i \right\|_{L^2(\mathbb{R}^4)} \left(\left\| \nabla u_i \right\|_{L^4(\mathbb{R}^4)} + \left\| \nabla u_i \right\|_{L^4(\mathbb{R}^4)}^2 \right). \end{aligned}$$

Similar to III , we can estimate IV by

$$\|IV\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C \left\| \nabla^2 u_i \right\|_{L^2(\mathbb{R}^4)} \left(\left\| \nabla^2 u_i \right\|_{L^2(\mathbb{R}^4)} + \left\| \nabla u_i \right\|_{L^4(\mathbb{R}^4)} + \left\| \nabla u_i \right\|_{L^4(\mathbb{R}^4)}^2 \right).$$

Putting the estimates of I, II, III, IV together, we have

$$\begin{aligned} \left\| \nabla^2 v_i \right\|_{L^{2,\infty}(\mathbb{R}^4)} &\leq C \left[\left\| h_i \right\|_{L^1(\mathbb{R}^4)} + \left\| \nabla^2 u_i \right\|_{L^2(\mathbb{R}^4)}^2 \right. \\ &\quad \left. + \left\| \nabla^2 u_i \right\|_{L^2(\mathbb{R}^4)} \left[\left\| \nabla u_i \right\|_{L^4(\mathbb{R}^4)} + \left\| \nabla u_i \right\|_{L^4(\mathbb{R}^4)}^2 \right] \right] \\ &\leq C \left[\left\| h \right\|_{L^1(\mathcal{B}_i)} + \left\| \nabla^2 u \right\|_{L^2(\mathcal{B}_i)}^{\frac{3}{2}} + \left\| \nabla^2 u \right\|_{L^2(\mathcal{B}_i)}^2 \right]. \end{aligned} \quad (4.12)$$

This clearly yields (4.9), since $\|\nabla^2 u\|_{L^2(\mathcal{B}_i)} \leq \epsilon \leq 1$.

Since w_i is a biharmonic function, the standard estimate and (4.12) imply that $w_i \in C^\infty(\mathcal{B}_i)$, and

$$\begin{aligned} \sup_{x \in \mathcal{A}_i} \left\{ |x|^2 \left| \nabla^2 w_i \right| (x) \right\} &\leq C \left\| \nabla^2 w_i \right\|_{L^{2,\infty}(\mathcal{B}_i)} \leq C \left[\left\| \nabla^2 u \right\|_{L^2(\mathcal{B}_i)} + \left\| \nabla^2 v_i \right\|_{L^{2,\infty}(\mathcal{B}_i)} \right] \\ &\leq C \left[\left\| h \right\|_{L^1(\mathcal{B}_i)} + \left\| \nabla^2 u \right\|_{L^2(\mathcal{B}_i)} \right]. \end{aligned} \quad (4.13)$$

This yields (4.10).

It follows from the above Claim that

$$\begin{aligned} &\sup_{\lambda \geq 1} \lambda^2 \sum_{i=0}^{K-1} \left| \left\{ x \in B_{2^{i+1}} \sqrt{\frac{\epsilon}{\lambda}} \setminus B_{2^i} \sqrt{\frac{\epsilon}{\lambda}} : |\nabla^2 u(x)| > \lambda \right\} \right| \\ &\leq \sup_{\lambda \geq 1} \lambda^2 \sum_{i=0}^{K-1} \left| \left\{ x \in \mathcal{A}_i : |\nabla^2 v_i(x)| > \frac{\lambda}{2} \right\} \right| + \sup_{\lambda \geq 1} \lambda^2 \sum_{i=0}^{K-1} \left| \left\{ x \in \mathcal{A}_i : |\nabla^2 w_i(x)| > \frac{\lambda}{2} \right\} \right| \\ &\leq C \sum_{i=0}^{K-1} \left\| \nabla^2 v_i \right\|_{L^{2,\infty}(\mathcal{A}_i)}^2 + C \sup_{\lambda \geq 1} \lambda^{-2} \sum_{i=0}^{K-1} \int_{\mathcal{A}_i} |\nabla^2 w_i|^4 dx \\ &\leq C \sum_{i=0}^{K-1} \left\| \nabla^2 v_i \right\|_{L^{2,\infty}(\mathcal{A}_i)}^2 + C \sup_{\lambda \geq 1} \lambda^{-2} \sum_{i=0}^{K-1} |\mathcal{A}_i| \left\| \nabla^2 w_i \right\|_{L^\infty(\mathcal{A}_i)}^4 \\ &\leq C \sum_{i=0}^{K-1} \left(\|h\|_{L^1(\mathcal{B}_i)}^2 + \|\nabla^2 u\|_{L^2(\mathcal{B}_i)}^3 \right) \\ &+ C \left[\sum_{i=0}^{K-1} (2^i \sqrt{\epsilon})^{-4} \right] \cdot \sup_{0 \leq i \leq K-1} [\|h\|_{L^1(\mathcal{B}_i)}^4 + \|\nabla^2 u\|_{L^2(\mathcal{B}_i)}^4] \\ &= I + II. \end{aligned}$$

Since Hölder inequality implies

$$\left\| h \right\|_{L^1(\mathcal{B}_i)} \leq C \delta^{4(1-\frac{1}{p})} \left\| h \right\|_{L^p(B_{2\delta})},$$

we have

$$\sup_{0 \leq i \leq K-1} \left[\|h\|_{L^1(\mathcal{B}_i)}^4 + \|\nabla^2 u\|_{L^2(\mathcal{B}_i)}^4 \right] \leq \epsilon^4 + C \left\| h \right\|_{L^p(B_{2\delta})}^4 \delta^{16(1-\frac{1}{p})}.$$

It is easy to see

$$\begin{aligned} I &\leq C \left[\sup_{0 \leq i \leq K-1} \|h\|_{L^1(\mathcal{B}_i)} \right] \int_{\cup_{i=0}^{K-1} \mathcal{B}_i} |h| + C \epsilon \int_{\cup_{i=0}^{K-1} \mathcal{B}_i} |\nabla^2 u|^2 \\ &\leq C \delta^{4(1-\frac{1}{p})} \|h\|_{L^p(B_{2\delta})} \|h\|_{L^1(B_{2\delta})} + C \epsilon \|\nabla^2 u\|_{L^2(B_{2\delta})}^2 \\ &\leq C \delta^{8(1-\frac{1}{p})} \|h\|_{L^p(B_{2\delta})}^2 + C \epsilon \|\nabla^2 u\|_{L^2(B_{2\delta})}^2, \end{aligned}$$

and

$$\begin{aligned} II &\leq C \epsilon^{-2} \sup_{0 \leq i \leq K-1} \left[\|h\|_{L^1(\mathcal{B}_i)}^4 + \|\nabla^2 u\|_{L^2(\mathcal{B}_i)}^4 \right] \\ &\leq C \left(\epsilon^2 + \epsilon^{-2} \delta^{16(1-\frac{1}{p})} \left\| h \right\|_{L^p(B_{2\delta})}^4 \right). \end{aligned}$$

Putting these estimates together we can obtain

$$\begin{aligned} \left\| \nabla^2 u \right\|_{L^{2,\infty}(B_\delta \setminus B_{2Rr})}^2 &\leq C \left(\epsilon^2 + \epsilon \left\| \nabla^2 u \right\|_{L^2(B_{2\delta})}^2 + \delta^4 + \delta^{8(1-\frac{1}{p})} \left\| h \right\|_{L^p(B_{2\delta})}^2 \right. \\ &\quad \left. + \epsilon^{-2} \delta^{16(1-\frac{1}{p})} \left\| h \right\|_{L^p(B_{2\delta})}^4 \right). \end{aligned}$$

This clearly implies (4.2). The proof is complete. \square

5 Proof of Theorem 1.2 and 1.3

This section is devoted to the proof of “=” of (1.10). The argument is based on the duality between $L^{2,1}$ and $L^{2,\infty}$.

Competition of Proof of Theorem 1.2:

For simplicity, we may assume $\Sigma = \{0\} \subset \Omega$ is a single point. In particular, $u_m \rightarrow u$ in $W_{\text{loc}}^{2,2}(B_{r_1} \setminus \{0\})$ for some $r_1 > 0$. By an induction argument similar to that of [3] in the context of harmonic maps, we may assume that there is only one bubble in B_{r_1} , i.e. $L_1 = 1$. Then for any $\epsilon > 0$, there exist $r_m \downarrow 0$, $R \geq 1$ sufficiently large, and $0 < \delta \leq \epsilon^{\frac{p}{4(p-1)}}$ such that for m sufficiently large, it holds

$$\int_{B_{2\rho} \setminus B_\rho} |\nabla^2 u_m|^2 \leq \epsilon^2, \quad \forall Rr_m \leq \rho \leq 2\delta. \quad (5.1)$$

Therefore by Lemma 4.1 we have

$$\begin{aligned} \left\| \nabla^2 u_m \right\|_{L^{2,\infty}(B_\delta \setminus B_{2Rr_m})} &\leq C \left[\epsilon + \delta^2 + \delta^{4(1-\frac{1}{p})} \left\| h_m \right\|_{L^p(B_{2\delta})} + \epsilon^{\frac{1}{2}} \left\| \nabla^2 u_m \right\|_{L^2(B_{2\delta})} \right. \\ &\quad \left. + \epsilon \left\| h_m \right\|_{L^p(B_{2\delta})}^2 \right] \\ &\leq C \epsilon^{\frac{1}{2}}. \end{aligned} \quad (5.2)$$

On the other hand, by Theorem 2.3 we have

$$\left\| \nabla^2 u_m \right\|_{L^{2,1}(B_\delta)} \leq C \left(\left\| \nabla^2 u_m \right\|_{L^2(B_{2\delta})} + \delta^{4(1-\frac{1}{p})} \left\| h_m \right\|_{L^p(B_{2\delta})} \right) \leq C. \quad (5.3)$$

Therefore by the duality between $L^{2,1}$ and $L^{2,\infty}$, we have

$$\begin{aligned} \left\| \nabla^2 u_m \right\|_{L^2(B_\delta \setminus B_{2Rr_m})}^2 &\leq C \left\| \nabla^2 u_m \right\|_{L^{2,1}(B_\delta \setminus B_{2Rr_m})} \left\| \nabla^2 u_m \right\|_{L^{2,\infty}(B_\delta \setminus B_{2Rr_m})} \\ &\leq C \epsilon^{\frac{1}{2}}. \end{aligned} \quad (5.4)$$

Since $\epsilon > 0$ is arbitrary, this yields that (1.10) holds. It is easy to see that (1.11) follows from (1.10) and the pointwise inequality $|\nabla u_m|^2 \leq |\Delta u_m|$. The proof of Theorem 1.2 is now complete. \square

Remark 5.1 *Under the same assumption as Theorem 1.2 and notations as above, it holds for any $x_0 \in \Sigma$,*

$$\lim_{\delta \downarrow 0} \lim_{R \uparrow \infty} \lim_{m \rightarrow \infty} \text{osc}_{B_{\frac{\delta}{2}}(x_0) \setminus B_{2Rr_m}(x_0)} u_m = 0. \quad (5.5)$$

Proof. By (5.4), we have that for any $\epsilon > 0$ there are $R > 1$ sufficiently large, $\delta > 0$ sufficiently small, and $m \geq 1$ sufficiently large so that (5.4) holds. By Fubini's theorem, we may assume

$$Rr_m \int_{\partial B_{2Rr_m}(x_0)} |\nabla^2 u_m|^2 \leq C\epsilon^{\frac{1}{2}}.$$

Let $v_m : B_{2Rr_m} \rightarrow \mathbb{S}^k$ be a minimizing biharmonic map extension of u_m such that $(v_m, \nabla v_m) = (u_m, \nabla u_m)$ on $\partial B_{2Rr_m}(x_0)$. Then we would have (cf. [17])

$$\int_{B_{2Rr_m}(x_0)} |\nabla^2 v_m|^2 \leq CRr_m \int_{\partial B_{2Rr_m}(x_0)} |\nabla^2 u_m|^2 \leq C\epsilon^{\frac{1}{2}}.$$

Now we define $w_m : B_\delta(x_0) \rightarrow \mathbb{S}^k$ by letting

$$w_m = \begin{cases} v_m & \text{in } B_{2Rr_m}(x_0) \\ u_m & \text{in } B_\delta(x_0) \setminus B_{2Rr_m}(x_0). \end{cases}$$

Then w_m is an approximate biharmonic map to \mathbb{S}^k with bi-tension field \widetilde{h}_m given by

$$\widetilde{h}_m = \begin{cases} 0 & \text{in } B_{2Rr_m}(x_0) \\ h_m & \text{in } B_\delta(x_0) \setminus B_{2Rr_m}(x_0). \end{cases}$$

Therefore by Theorem 2.3 we have

$$\left\| \nabla^2 w_m \right\|_{L^{2,1}(B_\delta(x_0))}^2 \leq C\epsilon^{\frac{1}{2}}.$$

Since there exists a harmonic function \widetilde{w}_m on $B_\delta(x_0)$, with $\|\nabla^2 \widetilde{w}_m\|_{L^2(B_\delta(x_0))} \leq C\epsilon^{\frac{1}{2}}$, such that

$$w_m(x) = \int_{B_\delta(x_0)} |x - y|^{-2} \Delta w_m(y) dy + \widetilde{w}_m(x), \quad x \in B_\delta(x_0).$$

It is easy to see

$$\text{osc}_{B_{\frac{\delta}{2}}(x_0)} \widetilde{w}_m \leq C\epsilon^{\frac{1}{2}},$$

while by the duality between $L^{2,1}$ and $L^{2,\infty}$ we have

$$\begin{aligned} & \left| \int_{B_\delta(x_0)} |x - y|^{-2} \Delta w_m(y) dy \right|_{L^\infty(B_{\frac{\delta}{2}}(x_0))} \\ & \leq C \left\| \frac{1}{|x|^2} \right\|_{L^{2,\infty}(B_1)} \left\| \nabla^2 w_m \right\|_{L^{2,1}(B_\delta(x_0))} \leq C\epsilon^{\frac{1}{4}}. \end{aligned}$$

Thus (5.5) follows. \square

Proof of Theorem 1.3:

It follows from the energy inequality (1.8) that there exists $t_m \uparrow +\infty$ such that $u_m(\cdot) = u(\cdot, t_m)$ is an approximate biharmonic map into \mathbb{S}^k with bi-tension fields $h_m = u_t(\cdot, t_m) \in L^2(\Omega)$ satisfying

$$\left\| h_m \right\|_{L^2(\Omega)} = \left\| u_t(\cdot, t_m) \right\|_{L^2(\Omega)} \rightarrow 0.$$

Moreover,

$$\left\| u_m \right\|_{W^{2,2}(\Omega)} \leq C \left\| u_0 \right\|_{W^{2,2}(\Omega)}.$$

Therefore we may assume that after taking another subsequence, $u_m \rightharpoonup u_\infty$ in $W^{2,2}(\Omega, \mathbb{S}^k)$. It is easy to see that u_∞ is a biharmonic map so that $u_\infty \in C^\infty(\Omega, \mathbb{S}^k)$ (see [20]). All other conclusions follow directly from Theorem 1.2. \square

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